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PLANE WAVES IN NON-SIMPLE ELASTIC SOLIDS

M. M. CARROLL

Division of Applied Mechanics, University of California, Berkeley

Abstract—A form of the constitutive equation, which is appropriate to describe the propagation of plane waves, of finite amplitude, in non-simple solids, is obtained. This equation is used to discuss the propagation of such waves in non-simple, holohedral isotropic, elastic solids of grade n. In particular, it is shown that a transverse, harmonic, circularly-polarized, plane progressive wave can propagate, in any such material, with phase velocity which depends on the wave amplitude and on the wavelength.

1. INTRODUCTION

GREEN and RIVLIN have developed a general nonlinear theory of non-simple continua, in which the force system consists of multipolar surface and body forces of various orders, whose rates of work are related to the velocity and the velocity gradients of various orders. A major difficulty in obtaining solutions of problems in this theory lies in the fact that the constitutive theory for non-simple materials has not been studied in any detail. Thus, reduced forms of the constitutive equations for such materials, with any particular symmetry, have not been obtained.

In this paper we show that the constitutive equations, which are appropriate to describe the propagation of plane waves (of finite amplitude) in non-simple materials, can be obtained quite readily. We confine our attention to non-simple elastic solids of grade n (for which the internal energy is a function of the deformation gradients of orders $1, 2, \ldots, n$), and especially to holohedral isotropic solids. In particular we show that a transverse, harmonic, circularly-polarized plane progressive wave can propagate in any such material, with phase velocity which depends on the wave amplitude and on the wavelength.

2. BASIC EQUATIONS

In this section we review the theory of Green and Rivlin [1] for a non-simple elastic material of grade n.

We refer the motion of a continuum to a fixed system of rectangular Cartesian axes and we express the coordinates x_i of a typical particle at time t as functions of its coordinates X_A, in an undeformed reference state, and of the time t, thus:

$$x_i = x_i(X_A, t). \tag{2.1}$$

Upper-case and lower-case Latin indices are associated with quantities pertaining to the reference state and to the deformed state, respectively, and take the values 1, 2, 3. A comma is used to denote partial differentiation with respect to the coordinates X_A or x_i, and the usual summation convention is used.

The internal energy U, per unit mass, for a non-simple elastic material of grade n is a single-valued function of the deformation gradients† of orders $1, 2, \ldots, n$, thus:

$$U = \hat{U}(x_{i,A_1\ldots A_\mu}) \qquad (\mu = 1, \ldots, n). \tag{2.2}$$

The invariance property of the internal energy under superposed rigid motions, i.e.,

$$\hat{U}(Q_{ij}x_{j,A_1\ldots A_\mu}) = \hat{U}(x_{i,A_1\ldots A_\mu}) \qquad (\mu = 1, \ldots, n) \tag{2.3}$$

for all proper orthogonal transformations Q_{ij}, implies the reduced form

$$U = \bar{U}(E_{AA_1\ldots A_\mu}) \qquad (\mu = 1, \ldots, n), \tag{2.4}$$

where $\bar{U}(\ldots)$ is a single-valued function of its arguments

$$E_{AA_1\ldots A_\mu} = x_{i,A}x_{i,A_1\ldots A_\mu} \qquad (\mu = 1, \ldots, n). \tag{2.5}$$

The form of the functions $\hat{U}(\ldots)$ and $\bar{U}(\ldots)$ is further restricted by any symmetries which the material may possess in the reference state, thus:

$$\hat{U}(Q_{A_1B_1}\ldots Q_{A_\mu B_\mu}x_{i,B_1\ldots B_\mu}) = \hat{U}(x_{i,A_1\ldots A_\mu}) \qquad (\mu = 1, \ldots, n) \tag{2.6}$$

or

$$\bar{U}(Q_{AB}Q_{A_1B_1}\ldots Q_{A_\mu B_\mu}E_{BB_1\ldots B_\mu}) = \bar{U}(E_{AA_1\ldots A_\mu}) \qquad (\mu = 1, \ldots, n) \tag{2.7}$$

for all transformations Q_{AB} in the group of material symmetry transformations. We shall restrict our attention to non-simple elastic solids, so that the material symmetry group is a (proper or improper) subgroup of the full orthogonal group of transformations.

Green and Rivlin [1] have given two alternative formulations of the theory of non-simple elastic solids. The following form of the basic equations is most convenient for our present purposes:

$$\pi_{Ai,A} + \rho_0 F_i = \rho_0 \ddot{x}_i, \tag{2.8}$$

$$\pi_{(A_1\ldots A_\mu)i} + \rho_0 F_{A_1\ldots A_\mu i} + \pi_{AA_1\ldots A_\mu i,A} = \rho_0 \frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_\mu}} \qquad (\mu = 1, \ldots, n-1), \tag{2.9}$$

$$\pi_{(A_1\ldots A_n)i} = \rho_0 \frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_n}}, \tag{2.10}$$

where $\pi_{(A_1\ldots A_\mu)i}$ $(\mu = 1, \ldots, n)$ is that part of $\pi_{A_1\ldots A_\mu i}$ which is completely symmetric in the indices $A_1, \ldots, A_\mu$. Here ρ_0 denotes the density of the material in the reference state, the monopolar and multipolar stress tensors π_{Ai} and $\pi_{A_1\ldots A_\mu i}$ are associated with monopolar and multipolar tractions across surfaces in the deformed state which were coordinate surfaces in the reference state, and are measured per unit area in the reference state, and F_i and $F_{A_1\ldots A_\mu i}$ are monopolar and multipolar body forces per unit mass.‡ We restrict our attention to the case when monopolar and multipolar body forces are absent. Substitution from equations (2.9) and (2.10) in the classical equations of motion (2.8) then gives

$$\sum_{\mu=1}^{n} (-1)^{\mu+1} \left\{ \frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_\mu}} \right\}_{,A_1\ldots A_\mu} = \ddot{x}_i. \tag{2.11}$$

† For simplicity, we restrict our attention to purely mechanical effects. We also consider only homogeneous materials.

‡ We assume that there are no multipolar inertia terms. (See Section 5.)

3. PLANE WAVES

In order to apply the theory of non-simple elastic solids, outlined in the previous section, to the solution of problems involving finite deformations of such materials, we must first determine the effect of the material symmetry condition (2.7) in restricting the form of the internal energy function $\bar{U}(\ldots)$. In general, this requires the calculation of an integrity basis, under the appropriate group of symmetry transformations, for n tensors of orders $2, \ldots, n+1$. Integrity bases, under various groups of symmetry transformations, have been calculated for vectors and second-order tensors† but not for tensors of higher order. We do not deal with this general problem in the present paper. Instead we effect a considerable simplification by restricting our attention to a very special class of deformations.

We consider the motion described by

$$x_i = \delta_{iA} X_A + u_i(Z, t), \qquad Z = X_A N_A, \tag{3.1}$$

where δ_{iA} denotes the Kronecker delta and N_A are the components of a constant unit vector ($N_A N_A = 1$). The deformation gradients associated with the motion (3.1) are given by

$$x_{i,A} = \delta_{iA} + u_i^{(1)} N_A, \tag{3.2}$$

$$x_{i,A_1\ldots A_\mu} = u_i^{(\mu)} N_{A_1} \ldots N_{A_\mu} \qquad (\mu = 2, \ldots, n), \tag{3.3}$$

where we have used the notation

$$u_i^{(\mu)} = \frac{\partial^\mu u_i}{\partial Z^\mu} \qquad (\mu = 1, \ldots, n). \tag{3.4}$$

It follows from equations (2.5), (3.2) and (3.3) that

$$E_{AA_1} = \delta_{AA_1} + u_A^{(1)} N_{A_1} + u_{A_1}^{(1)} N_A + u_i^{(1)} u_i^{(1)} N_A N_{A_1}, \tag{3.5}$$

$$E_{AA_1\ldots A_\mu} = (u_A^{(\mu)} + u_i^{(1)} u_i^{(\mu)} N_A) N_{A_1} \ldots N_{A_\mu} \qquad (\mu = 2, \ldots, n). \tag{3.6}$$

The internal energy U associated with the plane wave (3.1) is thus a single-valued function of the $n+1$ vectors $u_A^{(\mu)}$ ($\mu = 1, \ldots, n$) and N_A, thus:

$$U = \tilde{U}(u_A^{(\mu)}, N_A) \qquad (\mu = 1, \ldots, n). \tag{3.7}$$

It follows from equations (2.7), (3.5) and (3.6) that the material symmetry restriction on the function $\tilde{U}(\ldots)$ is

$$\tilde{U}(Q_{AB} u_B^{(\mu)}, Q_{AB} N_B) = \tilde{U}(u_A^{(\mu)}, N_A) \qquad (\mu = 1, \ldots, n) \tag{3.8}$$

for all (orthogonal) transformations Q_{AB} in the group of material symmetry transformations. Consequently $\tilde{U}(\ldots)$ is a scalar invariant function of its $n+1$ argument vectors under this group of transformations. Standard tables of basic invariants of vectors have been obtained for isotropic and transversely isotropic materials and for the crystal classes (see Adkins [3] and Smith and Rivlin [4]).‡

The equations (3.2) and (3.3) can be inverted in the form

$$u_i^{(1)} = (x_{i,A} - \delta_{iA}) N_A, \tag{3.9}$$

$$u_i^{(\mu)} = x_{i,A_1\ldots A_\mu} N_{A_1} \ldots N_{A_\mu} \qquad (\mu = 2, \ldots, n), \tag{3.10}$$

† See, for example, Wineman and Pipkin [2].

‡ We remark that the permissible forms of the function $\tilde{U}(\ldots)$ are restricted by equations (3.5) and (3.6). Thus, for example, the total degree of $\tilde{U}(\ldots)$ in $u_i^{(\mu)}$ ($\mu = 2, \ldots, n$) cannot exceed its total degree in $u_i^{(1)}$.

and hence we have

$$\frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_\mu}} = \frac{\partial \tilde{U}}{\partial u_i^{(\mu)}} N_{A_1} \ldots N_{A_\mu} \qquad (\mu = 1, \ldots, n). \tag{3.11}$$

Substitution from equation (3.11) in equation (2.11) yields the equation which governs the propagation of plane waves in a non-simple elastic material of grade n:

$$\sum_{\mu=1}^{n} (-1)^{\mu+1} \frac{\partial^\mu}{\partial Z^\mu} \left\{ \frac{\partial \tilde{U}}{\partial u_i^{(\mu)}} \right\} = \ddot{u}_i. \tag{3.12}$$

4. ISOTROPIC MATERIALS

We now consider the propagation of a plane wave of the form (3.1) in a non-simple elastic material of grade n, which is isotropic with a center of symmetry (holohedral isotropic) in the reference state. The group of material symmetry transformations for such a material is the full orthogonal group, and the condition (3.8) implies the reduced form

$$U = \overset{*}{U}(I_{\alpha\beta}, I_\gamma) \qquad (\alpha, \beta, \gamma = 1, \ldots, n) \tag{4.1}$$

for the internal energy, where

$$I_{\alpha\beta} = u_A^{(\alpha)} u_A^{(\beta)}, \qquad I_\gamma = u_A^{(\gamma)} N_A. \tag{4.2}$$

We may assume, without loss in generality, that the wave propagates in the X_3-direction, so that $N_A = \delta_{3A}$. Since $I_{\alpha\beta}$ is symmetric in its indices, we may also assume that $\overset{*}{U}(\ldots)$ is written as a symmetric function in $I_{\alpha\beta}$ and $I_{\beta\alpha}$. It follows from (3.7), (4.1) and (4.2) that

$$\frac{\partial \tilde{U}}{\partial u_i^{(\mu)}} = 2 \sum_{\lambda=1}^{n} \frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}} u_i^{(\lambda)} + \frac{\partial \overset{*}{U}}{\partial I_\mu} \delta_{3i}, \tag{4.3}$$

and substitution from (4.3) in the equations of motion (3.12) yields

$$\sum_{\mu=1}^{n} (-1)^{\mu+1} \frac{\partial^\mu}{\partial Z^\mu} \left\{ 2 \sum_{\lambda=1}^{n} \frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}} u_i^{(\lambda)} + \frac{\partial \overset{*}{U}}{\partial I_\mu} \delta_{3i} \right\} = \ddot{u}_i. \tag{4.4}$$

A. *Longitudinal waves*

A longitudinal plane wave is described by

$$x_1 = X_1, \qquad x_2 = X_2, \qquad x_3 = X_3 + u(Z, t) \qquad (Z = X_3). \tag{4.5}$$

For such a wave, the equations of motion (4.4) are satisfied identically for $i = 1, 2$, and for $i = 3$ we obtain

$$\sum_{\mu=1}^{n} (-1)^{\mu+1} \frac{\partial^\mu}{\partial Z^\mu} \left\{ 2 \sum_{\lambda=1}^{n} \frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}} \frac{\partial^\lambda u}{\partial Z^\lambda} + \frac{\partial \overset{*}{U}}{\partial I_\mu} \right\} = \ddot{u}, \tag{4.6}$$

where

$$I_{\lambda\mu} = I_\lambda I_\mu, \qquad I_\lambda = \frac{\partial^\lambda u}{\partial Z^\lambda}. \tag{4.7}$$

Thus, the propagation of longitudinal plane waves of finite amplitude is governed by the scalar equation (4.6). This equation may also be written in a more concise form similar to equation (3.12), and in general it is a nonlinear partial differential equation of order $2n$.

B. *Transverse waves*

A transverse plane wave is described by

$$x_1 = X_1 + u(Z, t), \qquad x_2 = X_2 + v(Z, t), \qquad x_3 = X_3 = Z. \tag{4.8}$$

For such a wave, the equations of motion (4.4) become

$$\begin{aligned} 2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{\mu+1}\frac{\partial^\mu}{\partial Z^\mu}\left\{\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}\frac{\partial^\lambda u}{\partial Z^\lambda}\right\} &= \ddot{u}, \\ 2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{\mu+1}\frac{\partial^\mu}{\partial Z^\mu}\left\{\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}\frac{\partial^\lambda v}{\partial Z^\lambda}\right\} &= \ddot{v}, \\ \sum_{\mu=1}^{n}(-1)^{\mu+1}\frac{\partial^\mu}{\partial Z^\mu}\left\{\frac{\partial \overset{*}{U}}{\partial I_{\mu}}\right\} &= 0, \end{aligned} \tag{4.9}$$

where

$$I_{\lambda\mu} = \frac{\partial^\lambda u}{\partial Z^\lambda}\frac{\partial^\mu u}{\partial Z^\mu} + \frac{\partial^\lambda v}{\partial Z^\lambda}\frac{\partial^\mu v}{\partial Z^\mu}, \qquad I_\mu = 0. \tag{4.10}$$

It is evident that, in general, solutions u and v of equations $(4.9)_{1,2}$ will not satisfy equation $(4.9)_3$. Thus, for example, a linearly-polarized transverse wave ($v \equiv 0$, say) cannot propagate in a general isotropic non-simple elastic solid. However, if the material is incompressible, then it can be shown, by appropriate modification of the theory of Green and Rivlin, that the equations of motion (2.11) become

$$-p_{,i} + \sum_{\mu=1}^{n}(-1)^{\mu+1}\left\{\frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_\mu}}\right\}_{,A_1\ldots A_\mu} = \ddot{x}_i, \tag{4.11}$$

where p is an arbitrary scalar function. This implies a similar modification of the equations (4.9), and the function p can be chosen so that equation $(4.9)_3$ is satisfied for any solution u, v of equations $(4.9)_{1,2}$.

We now consider the special case of a transverse circularly-polarized harmonic plane progressive wave, described by equation (4.8), with

$$u(Z, t) = a \cos k(Z - ct), \qquad v(Z, t) = a \sin k(Z - ct), \tag{4.12}$$

where a, k and c are constants. It has been shown [5] that such a wave can propagate in every isotropic simple elastic solid, with phase velocity c which depends on the amplitude a. Differentiation of equation (4.12) yields

$$\begin{aligned} \frac{\partial^{2\lambda} u}{\partial Z^{2\lambda}} &= (-1)^\lambda k^{2\lambda} u, & \frac{\partial^{2\lambda+1} u}{\partial Z^{2\lambda+1}} &= (-1)^{\lambda+1} k^{2\lambda+1} v, \\ \frac{\partial^{2\lambda} v}{\partial Z^{2\lambda}} &= (-1)^\lambda k^{2\lambda} v, & \frac{\partial^{2\lambda+1} v}{\partial Z^{2\lambda+1}} &= (-1)^{\lambda} k^{2\lambda+1} u, \end{aligned} \qquad (\lambda = 0, 1, \ldots) \tag{4.13}$$

and substitution from these equations in equations (4.10) yields

$$\begin{aligned} I_{\lambda\mu} &= (-1)^{(\lambda-\mu)/2} k^{\lambda+\mu} a^2 \quad \text{for } \lambda+\mu \text{ even}, \\ &= 0 \qquad\qquad\qquad\quad\ \text{for } \lambda+\mu \text{ odd}. \end{aligned} \tag{4.14}$$

Thus, the isotropic invariants associated with the motion (4.8), (4.12) are independent of position and time, so that the internal energy is also constant. The equations of motion $(4.9)_{1,2}$ become

$$\begin{aligned} 2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{\mu+1}\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}\frac{\partial^{\lambda+\mu}u}{\partial Z^{\lambda+\mu}} &= \ddot{u}, \\ 2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{\mu+1}\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}\frac{\partial^{\lambda+\mu}v}{\partial Z^{\lambda+\mu}} &= \ddot{v}, \end{aligned} \tag{4.15}$$

and equation $(4.9)_3$ is satisfied identically. It is evident that the terms in equations (4.15) for which $\lambda+\mu$ is odd will cancel in pairs, due to the factor $(-1)^{\mu+1}$ and the symmetry of $I_{\lambda\mu}$. Consequently, equations (4.12) and (4.15) are compatible provided†

$$c^2 = 2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{(3\mu+\lambda)/2}\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}k^{\lambda+\mu-2}, \tag{4.16}$$

with $I_{\lambda\mu}$ given by (4.14), where we need only consider those terms in (4.16) for which $\lambda+\mu$ is even. Thus, a transverse, circularly-polarized harmonic plane progressive wave can propagate in any (compressible or incompressible) isotropic non-simple elastic solid of grade n, with phase velocity c which is given by the dispersion relation (4.16) and which depends both on the amplitude a and on the wavelength $2\pi/k$.

5. MULTIPOLAR INERTIA TERMS

In writing the equation of motion (2.11) we assumed that all multipolar inertia terms may be neglected. We now consider briefly a more general theory (see Green and Rivlin [7]) in which the kinetic energy per unit mass K is given by

$$\begin{aligned} 2K = v_i v_i &+ \sum_{\alpha=1}^{n-1} b_{A_1\ldots A_\alpha} v_i v_{i,A_1\ldots A_\alpha} \\ &+ \sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n-1} c_{A_1\ldots A_\alpha:B_1\ldots B_\beta} v_{i,A_1\ldots A_\alpha} v_{i,B_1\ldots B_\beta}, \end{aligned} \tag{5.1}$$

where $b_{A_1\ldots A_\alpha}$ and $c_{A_1\ldots A_\alpha:B_1\ldots B_\beta}$ are constants. Without loss in generality we may assume that the $b_{A_1\ldots A_\alpha}$ are completely symmetric in the indices $A_1\ldots A_\alpha$ and the $c_{A_1\ldots A_\alpha:B_1\ldots B_\beta}$ are completely symmetric in the indices $A_1\ldots A_\alpha$ and $B_1\ldots B_\beta$, and satisfy

$$c_{A_1\ldots A_\alpha:B_1\ldots B_\beta} = c_{B_1\ldots B_\beta:A_1\ldots A_\alpha}. \tag{5.2}$$

† We assume that a and k are such that equation (4.16) yields a positive value of c^2.

The monopolar and multipolar inertial forces per unit mass, obtained from equation (5.1), are

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial v_i}\right) = -\dot{v}_i - \frac{1}{2}\sum_{\alpha=1}^{n-1} b_{A_1\ldots A_\alpha}\dot{v}_{i,A_1\ldots A_\alpha} \tag{5.3}$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial v_{i,A_1\ldots A_\alpha}}\right) = -\tfrac{1}{2}b_{A_1\ldots A_\alpha}\dot{v}_i - \sum_{\beta=1}^{n-1} c_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\dot{v}_{i,B_1\ldots B_\beta}. \tag{5.4}$$

We see from equations (2.8), (2.9), (2.10), (5.3) and (5.4) that the equation of motion (2.11) must now be replaced by

$$\begin{aligned}\sum_{\mu=1}^{n}(-1)^{\mu+1}\left\{\frac{\partial \hat{U}}{\partial x_{i,A_1\ldots A_\mu}}\right\}_{,A_1\ldots A_\mu} &= \dot{v}_i + \frac{1}{2}\sum_{\gamma=1}^{n-1}\{1+(-1)^\gamma\}b_{A_1\ldots A_\gamma}\dot{v}_{i,A_1\ldots A_\gamma} \\ &\quad + \sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n-1}(-1)^\alpha c_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\dot{v}_{i,A_1\ldots A_\alpha B_1\ldots B_\beta}.\end{aligned} \tag{5.5}$$

We remark that the terms for which γ or $\alpha+\beta$ are odd do not contribute to the expression on the right-hand side of equation (5.5). Thus, in particular, the transverse harmonic, circularly-polarized, plane progressive wave, described in the previous section, can propagate in an isotropic material. The dispersion relation (4.16) must now be replaced by

$$\begin{aligned}2\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}(-1)^{(3\mu+\lambda)/2}\frac{\partial \overset{*}{U}}{\partial I_{\lambda\mu}}k^{\lambda+\mu-2} &= c^2\Bigg\{1+\sum_{\gamma=1}^{n-1}(-1)^{\gamma/2}k^\gamma b_{A_1\ldots A_\gamma}N_{A_1}\ldots N_{A_\gamma} \\ &\quad + \sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n-1}(-1)^{(3\alpha+\beta)/2}k^{\alpha+\beta}c_{A_1\ldots A_\alpha:B_1\ldots B_\beta}N_{A_1}\ldots N_{A_\alpha}N_{B_1}\ldots N_{B_\beta}\Bigg\},\end{aligned} \tag{5.6}$$

where N_A are the components of a unit vector in the direction of propagation† and we need only consider those terms in (5.6) for which $\lambda+\mu$, γ and $\alpha+\beta$ are even.

6. REMARKS

The results presented here may be generalized in various ways. For example, the motion (3.1) may be extended to include a homogeneous deformation, thus:

$$x_i = F_{iA}X_A + u_i(Z, t), \tag{6.1}$$

where the F_{iA} are constants. The material symmetry condition then necessitates the calculation of integrity basis for $n+1$ vectors and one second-order tensor. Again, the material symmetry considerations may be applied, in an obvious way, to non-simple materials with memory (see Green and Rivlin [6]). Finally, it was shown in [5] that a circularly-polarized wave of the form (4.8), (4.12) can propagate in the symmetry direction of a transversely-isotropic simple elastic solid, which may also be subjected to uniform stretching in the symmetry direction. A similar extension is possible here. However, the more

† If the form (5.1) of the kinetic energy K is not isotropic, then the phase velocity c will depend on the direction of propagation of the wave.

general wave described by equation (4.8), with

$$u(Z, t) = a \int^{Z-ct} \cos f(\varphi)\, d\varphi, \qquad v(Z, t) = a \int^{Z-ct} \sin f(\varphi)\, d\varphi, \tag{6.2}$$

where $f(\cdot)$ is an arbitrary function, can propagate in a simple elastic solid but not in a non-simple one.

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Абстракт—Предлагается форма определяющего уравнения, которая описывает распределение плоских волн с конечной амплитуды в непростых телах. Используется зто уравнение для обсуждения распределения зтих волн в непростых, голоздрических, изотропных, упругих телах, "n"-того порядка. В особенности, оказывается, что поперечная, гармоническая, кругло-поляризованная, плоская поступательная волна может распределятся, в любом таком материале, с фазовой скоростью, которая зависит от амплитуды и длины волны.